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# The conservation laws and integrals of motion for a certain class of equations in discrete models

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**Abstract.** We derive the conservation laws for a wide class of discrete equations, including the $q$-deformations of classical field-theoretic equations. The explicit expressions for the conserved currents are used in the construction of the integrals of motion. The procedure is applied to the nonlinear difference equation and to the $q$-wave equations in $D = 3$ and $D = 4$.

## 1. Introduction

We shall consider a simple and effective procedure for the construction of the deformed conservation law and integrals of motion for a class of equations that depend on generalized difference derivatives. Such equations occur in quantum algebra theory as the realizations of the eigenvalue problem for Casimir operators [1–3], in the deformations of field-theoretic equations, for example the $\kappa$-Klein–Gordon equation [1], the $\kappa$-Dirac equation [4], the $q$-heat equation and $q$-wave equations introduced in [3], as well as in discrete models [5–8]. Let us note that the application of the standard difference derivative to field theory with non-localized action, as well as to the relativistic model of the electron, has also been studied in the literature [9–11].

We propose to extend the formalism of standard difference derivatives and use the generalized difference derivative [12, 13]. This formulation allows us partly to extend some of the results of discrete mechanics [5–8] as we can include the lattice structure into the equations.

On the other hand, this formalism can easily be applied to discrete field models on an arbitrary lattice.

In this paper we focus on almost linear equations with only one nonlinear term independent of derivatives. As we know, for this class of equations we can derive, in classical theory, the conservation laws using the Takahashi–Umezawa procedure [14].

Our main result is therefore the extension of this method to models with generalized difference derivatives replacing the differential ones. The case of equations with symmetric generalized difference derivatives has been considered in [15]. We begin in section 2 with a brief review of the results of [15]. In section 3 we present the construction of the conservation laws and integrals of motion for models described by non-symmetric generalized difference derivatives. Then the method thus introduced is applied to the $q$-deformation of the wave equations and to the nonlinear general difference equation.

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Throughout the paper the following notation will be used.

- For the symmetric generalized difference derivative

$$\bar{\partial}_\phi f(t) := \frac{f(\phi(t)) - f(\phi^{-1}(t))}{\phi(t) - \phi^{-1}(t)}\,. \tag{1}$$

- For the non-symmetric generalized difference derivative

$$\partial_\phi f(t) = \frac{f(\phi(t)) - f(t)}{\phi(t) - t}\,. \tag{2}$$

- For the transformation operators

$$\zeta^+ f(t) = f(\phi(t)) \qquad \zeta^- f(t) = f(\phi^{-1}(t)) \qquad \zeta^n f(t) = f(\phi^n(t))\,. \tag{3}$$

The corresponding partial derivatives and transformation operators acting on the respective variables will be denoted as $\bar{\partial}_{\phi_\mu}$ , $\partial_{\phi_\mu}$ and $\zeta_\mu$.

We have chosen to number the points of the lattice by iterations of the transformation $\phi$. Nevertheless, all the results remain valid if one rewrites them according to standard numbering. The notation is then as follows:

- For the symmetric generalized difference derivative:

$$\bar{\partial}_\phi f(t_n) := \frac{f(t_{n+1}) - f(t_{n-1})}{t_{n+1} - t_{n-1}}\,.$$

- For the non-symmetric generalized difference derivative:

$$\partial_\phi f(t_n) = \frac{f(t_{n+1}) - f(t_n)}{t_{n+1} - t_n}\,.$$

- For the transformation operators:

$$\zeta^+ f(t_n) = f(t_{n+1}) \qquad \zeta^- f(t_n) = f(t_{n-1}) \qquad \zeta^m f(t_n) = f(t_{n+m})\,.$$

## 2. The conservation laws and integrals of motion in the models described by symmetric generalized difference derivatives

Let us begin with a brief review of the results of [15], in which we derived the conservation laws for linear equations with a generalized symmetric difference derivative of the form

$$\Lambda(\bar{\partial})\Phi = 0 \tag{4}$$

$$\Lambda(\bar{\partial}) = \sum_{l=0}^{N} \Lambda_{\mu_1\ldots\mu_l} \bar{\partial}_{\phi_{\mu_1}} \ldots \bar{\partial}_{\phi_{\mu_l}} \tag{5}$$

where the coefficients (which may be matrices) are constant and symmetric with respect to the permutation of the set of indices $(\mu_1 \ldots \mu_l)$ for each $l$. The symmetry properties of the coefficients are due to the fact that the derivatives $\bar{\partial}_{\phi_\mu}$ commute.

The Leibnitz rule for this type of operator is deformed in comparison with the classical differential calculus:

$$\bar{\partial}_\phi [f \cdot g] = [\bar{\partial}_\phi f]\zeta^+ g + [\zeta^- f]\bar{\partial}_\phi g\,. \tag{6}$$

We have modified the Leibnitz rule so as to get on the right-hand side operators acting only on one of the functions in the product. Taking into account the modified Leibnitz rule for the symmetric derivative

$$\mathcal{D}_\phi [f(\zeta^- + \overleftarrow{\zeta}^-)g] = f(\bar{\partial}_\phi + \overleftarrow{\bar{\partial}}_\phi)g \tag{7}$$

where

$$\mathcal{D}_\phi f(t) := \frac{f(\phi(t)) - f(t)}{\phi(t) - \phi^{-1}(t)}$$

as well as the classical results of Takahashi and Umezawa [14] for linear equations, we now construct the operator $\Gamma_\mu$ with the property

$$\sum_\mu (\bar\partial_{\phi_\mu} + \overleftarrow{\bar\partial}_{\phi_\mu}) \circ \Gamma_\mu(\bar\partial, -\overleftarrow{\bar\partial}) = \Lambda(\bar\partial) - \Lambda(-\overleftarrow{\bar\partial})\,. \tag{8}$$

Here we have introduced the product sign $\circ$ to stress the way the left and right operators act on the corresponding sides in the following formulae. Straightforward calculation shows that

$$\Gamma_\mu(\bar\partial, -\overleftarrow{\bar\partial}) = \sum_{l=1}^{N-1} \sum_{k=0}^{l} \Lambda_{\mu\mu_1\ldots\mu_l} (-\overleftarrow{\bar\partial}_{\phi_{\mu_1}}) \ldots (-\overleftarrow{\bar\partial}_{\phi_{\mu_k}}) \bar\partial_{\phi_{\mu_{k+1}}} \ldots \bar\partial_{\phi_{\mu_l}} + \Lambda_\mu \tag{9}$$

fulfils the above equation (8).

This operator is also unique in the class of local operators. Let us sketch the proof.

*Proof.* Let us denote monomials of derivatives acting on the left- and right-hand sides as follows:

$$[\mu_1, \ldots, \mu_k] = \bar\partial_{\phi_{\mu_1}} \ldots \bar\partial_{\phi_{\mu_k}} \qquad \overline{[\mu_1, \ldots, \mu_k]} = (-\overleftarrow{\bar\partial}_{\phi_{\mu_1}}) \ldots (-\overleftarrow{\bar\partial}_{\phi_{\mu_k}})$$

It is clear that to solve equation (8) locally (that means in the case of discrete models to get the result depending on a finite number of non-negative powers of the derivatives) we should consider the polynomial solution in operators $\bar\partial$ and $\overleftarrow{\bar\partial}$ of order $N-1$. We must take into account two facts:

- Only the constant coefficients in such a polynomial are admissible, because the dependence of coefficients on variables would provide on the right-hand side of equality (8) terms depending on the operators $\zeta^+$ and $\zeta^-$ due to (6). These operators cannot be expressed as polynomials in derivatives and therefore do not appear in $\Lambda(\bar\partial) - \Lambda(-\overleftarrow{\bar\partial})$.
- In discrete models the left and right derivatives with respect to the same variable do not commute (in contrast to classical differential calculus). Nevertheless it is sufficient to consider mixed monomials with fixed ordering $\overline{[\mu_1, \ldots, \mu_k]}[\mu_{k+1}, \ldots, \mu_l]$, as mixed monomials with any other ordering would produce non-constant coefficients in (10) and, by the first remark, are not allowed.

Now we are able to write the general form of the solution for equation (8):

$$\Gamma_\mu(\bar\partial, -\overleftarrow{\bar\partial}) = \sum_{l=1}^{N-1} \sum_{k=0}^{l} a^k_{\mu\mu_1\ldots\mu_l} \overline{[\mu_1, \ldots, \mu_k]}[\mu_{k+1}, \mu_{k+2}, \ldots, \mu_l] + a^0_\mu \tag{10}$$

where the coefficients are constant.

We derive the explicit form of the coefficients in (10) using the condition (8):

$$\begin{aligned}
&\sum_\mu (\bar\partial_{\phi_\mu} + \overleftarrow{\bar\partial}_{\phi_\mu}) \circ \Gamma_\mu(\bar\partial, -\overleftarrow{\bar\partial}) \\
&\quad = \sum_{l=1}^{N-1} \sum_{k=0}^{l} a^k_{\mu\mu_1\ldots\mu_l} \left(-\overline{[\mu_1, \ldots, \mu_k, \mu]}[\mu_{k+1}, \mu_{k+2}, \ldots, \mu_l]\right. \\
&\qquad \left. + \overline{[\mu_1, \ldots, \mu_k]}[\mu, \mu_{k+1}, \mu_{k+2}, \ldots, \mu_l]\right) \\
&\qquad + a^0_\mu \left(-\overline{[\mu]} + [\mu]\right) = \Lambda(\bar\partial) - \Lambda(-\overleftarrow{\bar\partial})\,.
\end{aligned}$$

Comparing coefficients of monomials of the same type on the left- and right-hand sides we get the following equations for the coefficients:

$$a^0_{\mu\mu_1\ldots\mu_l} = \Lambda_{\mu\mu_1\ldots\mu_l} \qquad a^0_\mu = \Lambda_\mu \qquad 1 \leqslant l \leqslant N-1$$

and

$$-a^k_{\mu\mu_1\ldots\mu_l} + a^{k+1}_{\mu_{k+1}\mu_1\ldots\mu_k\mu\mu_{k+2}\ldots\mu_l} = 0 \qquad 0 \leqslant k \leqslant l-1 \qquad 1 \leqslant l \leqslant N-1\,. \tag{11}$$

Due to the symmetry of the coefficients $\Lambda$ the only solution of the system (11) is:

$$a^k_{\mu_1\ldots\mu_l} = \Lambda_{\mu_1\ldots\mu_l} \qquad 0 \leqslant k \leqslant l \qquad 0 \leqslant l \leqslant N$$

where all the coefficients of (10) are also symmetric with respect to permutations of lower indices. This unique solution yields the operator $\Gamma_\mu$ of the form determined in (9) and that concludes the proof. □

Now, having two solutions of the equation of motion

$$\Lambda(\bar\partial)G = F\Lambda(-\overleftarrow{\bar\partial}) = 0 \tag{12}$$

we are able to construct the current

$$J_\mu = F\hat\Gamma_\mu(\bar\partial, -\overleftarrow{\bar\partial})G \tag{13}$$

with

$$\hat\Gamma_\mu(\bar\partial, -\overleftarrow{\bar\partial}) = \sum_{l=1}^{N-1}\sum_{k=0}^{l} \Lambda_{\mu\mu_1\ldots\mu_l}(-\overleftarrow{\bar\partial}_{\phi_{\mu_1}})(-\overleftarrow{\bar\partial}_{\phi_{\mu_k}})(\zeta^-_\mu + \overleftarrow{\zeta}^-_\mu)\bar\partial_{\phi_{\mu_{k+1}}}\ldots\bar\partial_{\phi_{\mu_l}} + \Lambda_\mu(\zeta^-_\mu + \overleftarrow{\zeta}^-_\mu)$$

which obeys the deformed conservation law

$$\sum_\mu \mathcal{D}_{\phi_\mu} J_\mu = 0\,. \tag{14}$$

Let us recall the definite integrals introduced in [15] fulfilling the fundamental relations

$$\int_t^a \mathcal{D}_\phi u(\tau)\,\mathrm{d}\mu_\phi(\tau) = u(a) - u(t) \qquad \int_b^t \mathcal{D}_\phi u(\tau)\,\mathrm{d}\mu_\phi(\tau) = u(t) - u(b) \tag{15}$$

$$\mathcal{D}_\phi \int_t^a u(\tau)\,\mathrm{d}\mu_\phi(\tau) = -u(t) \qquad \mathcal{D}_\phi \int_b^t u(\tau)\,\mathrm{d}\mu_\phi(\tau) = u(t) \tag{16}$$

which are correctly defined when there exist (finite or not) the following limits:

$$\lim_{n\to\infty}\phi^n(t) = a \qquad \lim_{n\to\infty}\phi^{-n}(t) = b\,. \tag{17}$$

When only one of the limits exists we can use only one of these integrals in further construction. After integrating over the spatial variables we obtain the integrals of motion, which in the four-dimensional model appear as follows:

$$\begin{aligned} Q(x_0) &= \int \mathrm{d}\mu_{\phi_1}\,\mathrm{d}\mu_{\phi_2}\,\mathrm{d}\mu_{\phi_3}\; J_0(x_0, \boldsymbol{x}) \\ \mathcal{D}_{\phi_0} Q(x_0) &= -\int \mathrm{d}\mu_{\phi_1}\,\mathrm{d}\mu_{\phi_2}\,\mathrm{d}\mu_{\phi_3}\; \sum_k \mathcal{D}_{\phi_k} J_k = \text{boundary terms} = 0\,. \end{aligned} \tag{18}$$

This equation means that $Q$ is in fact constant on the time lattice.

## 3. Models with non-symmetric generalized difference derivatives

### 3.1. The conservation laws

Our aim is to derive the conservation laws for equations of motion depending on the generalized non-symmetric difference derivatives. We shall start as before from the Leibnitz rule which now appears as follows:

$$\partial_\phi f \cdot g(t) = \partial_\phi f(t)\zeta^+ g(t) + f(t)\partial_\phi g(t) \tag{19}$$

and leads to the equality

$$\partial_\phi f \cdot \zeta^- g(t) = g(t)\partial_\phi f(t) - f(t)\partial_\phi^\dagger g(t) = f\big[\overleftarrow{\partial}_\phi - \partial_\phi^\dagger\big]g(t) \tag{20}$$

where we have used the conjugate operator

$$\partial_\phi^\dagger = -\partial_\phi \zeta^- . \tag{21}$$

For the difference and $q$-derivative this operator is of the form

$$D_h^\dagger = -D_{-h} \qquad \partial_q^\dagger = -q^{-1}\partial_{q^{-1}} .$$

Let us now consider the equation of motion

$$\Lambda(\partial) = \sum_{l=0}^{N} \Lambda_{\mu_1 \ldots \mu_l} \partial_{\phi_{\mu_1}} \ldots \partial_{\phi_{\mu_l}} \tag{22}$$

where the coefficients (which may be matrices) are constant and symmetric with respect to the permutation of the set of indices $(\mu_1 \ldots \mu_l)$ for each $l$.

As we can see, the equation of motion depends on the partial generalized difference derivatives, but our construction applies to mixed models with difference and differential derivatives as well. All the formulae for mixed models in the symmetric and non-symmetric cases can be obtained by inserting the respective partial differential derivatives and taking into account the fact that the operator $\zeta$ with respect to the corresponding variable becomes identity and the discrete integral is then the continuous one. An example of such a model is the $\kappa$-Klein–Gordon equation [1] for which we have presented the conservation law and integrals of motion in [15], as well as the $\kappa$-Dirac equation [4].

Similarly to the previous section we construct the unique local (in the sense of section 2) operator $\Gamma_\mu$ fulfilling the equality

$$\sum_\mu (\overleftarrow{\partial}_{\phi_\mu} - \partial_{\phi_\mu}^\dagger) \circ \Gamma_\mu(\overleftarrow{\partial}, \partial^\dagger) = \Lambda(\overleftarrow{\partial}) - \Lambda(\partial^\dagger) \tag{23}$$

$$\Gamma_\mu(\overleftarrow{\partial}, \partial^\dagger) = \sum_{l=1}^{N-1} \sum_{k=0}^{l} \Lambda_{\mu\mu_1 \ldots \mu_l} \overleftarrow{\partial}_{\phi_{\mu_1}} \ldots \overleftarrow{\partial}_{\phi_{\mu_k}} \partial_{\phi_{\mu_{k+1}}}^\dagger \ldots \partial_{\phi_{\mu_l}}^\dagger + \Lambda_\mu . \tag{24}$$

Let us check this formula to show the meaning of the product $\circ$:

$$\begin{aligned}
&\sum_\mu (\overleftarrow{\partial}_{\phi_\mu} - \partial_{\phi_\mu}^\dagger) \circ \Gamma_\mu(\overleftarrow{\partial}, \partial^\dagger) \\
&\quad = \sum_{l=1}^{N-1} \sum_{k=0}^{l} \Lambda_{\mu\mu_1 \ldots \mu_l} \overleftarrow{\partial}_{\phi_{\mu_1}} \ldots \overleftarrow{\partial}_{\phi_{\mu_k}} (\overleftarrow{\partial}_{\phi_\mu} - \partial_{\phi_\mu}^\dagger)\partial_{\phi_{\mu_{k+1}}}^\dagger \ldots \partial_{\phi_{\mu_l}}^\dagger + \Lambda_\mu(\overleftarrow{\partial}_{\phi_\mu} - \partial_{\phi_\mu}^\dagger) \\
&\quad = \Lambda(\overleftarrow{\partial}) - \Lambda(\partial^\dagger) .
\end{aligned}$$

In the following proof we can see by construction that the operator $\Gamma_\mu$ given by (24) is unique in the class of local discrete operators.

*Proof.* As in section 2 within this proof we denote the monomials of derivatives acting on the right- and left-hand sides as follows:

$$[\mu_1, \ldots, \mu_k] = \partial^{\dagger}_{\phi_{\mu_1}} \ldots \partial^{\dagger}_{\phi_{\mu_k}} \qquad \overline{[\mu_1, \ldots, \mu_k]} = \overleftarrow{\partial}_{\phi_{\mu_1}} \ldots \overleftarrow{\partial}_{\phi_{\mu_k}} .$$

From the Leibnitz rule (19) for the non-symmetric derivative it is clear that we should consider now the general polynomial of order $N-1$ with functional coefficients. It has the following form:

$$\Gamma_\mu(\overleftarrow{\partial}, \partial^{\dagger}) = \sum_{l=1}^{N-1} \sum_{k=0}^{l} \overline{[\mu_1, \ldots, \mu_k]} a^k_{\mu\mu_1\ldots\mu_l} [\mu_{k+1}, \mu_{k+2}, \ldots, \mu_l] + a^0_\mu$$

We apply the condition (23) to the general form of the solution in order to derive the explicit formulae for the coefficients:

$$\begin{aligned}
&\sum_\mu (\overleftarrow{\partial}_{\phi_\mu} - \partial^{\dagger}_{\phi_\mu}) \circ \Gamma_\mu(\overleftarrow{\partial}, \partial^{\dagger}) \\
&\quad = \sum_{l=1}^{N-1} \sum_{k=0}^{l} \sum_\mu \overline{[\mu_1, \ldots, \mu_k, \mu]} a^k_{\mu\mu_1\ldots\mu_l} [\mu_{k+1}, \mu_{k+2}, \ldots, \mu_l] \\
&\qquad - \sum_{l=1}^{N} \sum_{k=0}^{l} \overline{[\mu_1, \ldots, \mu_k]} \sum_\mu \left(\zeta^-_\mu a^k_{\mu\mu_1\ldots\mu_l}\right) [\mu, \mu_{k+1}, \mu_{k+2}, \ldots, \mu_l] \\
&\qquad - \sum_{l=1}^{N} \sum_{k=0}^{l} \overline{[\mu_1, \ldots, \mu_k]} \sum_\mu \left(\partial^{\dagger}_{\phi_\mu} a^k_{\mu\mu_1\ldots\mu_l}\right) [\mu_{k+1}, \mu_{k+2}, \ldots, \mu_l] \\
&\qquad + \sum_\mu \overline{[\mu]} a^0_\mu - \sum_\mu (\zeta^-_\mu a^0_\mu)[\mu] - \sum_\mu (\partial^{\dagger}_{\phi_\mu} a^0_\mu) = \Lambda(\overleftarrow{\partial}) - \Lambda(\partial^{\dagger}) .
\end{aligned}$$

As in the proof from section 2, we compare the coefficients of monomials of the same type on both sides of the above condition and obtain the following set of difference equations for the functions $a^k_{\mu_1\ldots\mu_j}$ (no summation over repeating indices):

$$\zeta^-_\mu a^0_{\mu\mu_1\ldots\mu_l} = \Lambda_{\mu\mu_1\ldots\mu_l} \qquad \zeta^-_\mu a^0_\mu = \Lambda_\mu \qquad 1 \leqslant l \leqslant N-1 \tag{25}$$

and

$$a^k_{\mu\mu_1\ldots\mu_l} - \zeta^-_\mu a^{k+1}_{\mu_{k+1}\mu_1\ldots\mu_k\mu\mu_{k+2}\ldots\mu_l} - \sum_\alpha \partial^{\dagger}_{\phi_\alpha} a^k_{\alpha\mu_1\ldots\mu_k\mu\mu_{k+1}\ldots\mu_l} = 0 \tag{26}$$

where

$$0 \leqslant k \leqslant l-1 \qquad 1 \leqslant l \leqslant N-1 .$$

Starting from (25) we get the unique constant solution for the coefficients $a^0$:

$$a^0_{\mu_1\ldots\mu_l} = \Lambda_{\mu_1\ldots\mu_l} \qquad 1 \leqslant l \leqslant N .$$

This symmetric constant solution for initial coefficients allows us to evaluate the remaining ones, and we see that they are also constant and symmetric:

$$a^k_{\mu_1\ldots\mu_l} = \Lambda_{\mu_1\ldots\mu_l} \qquad 1 \leqslant l \leqslant N \quad 1 \leqslant k \leqslant l .$$

The derivation of the explicit formulae for the unique solution of the coefficients of the operator $\Gamma_\mu$ concludes the proof of the formula (24). □

Now we need two solutions of the corresponding equation of motion and its conjugate version:

$$G\Lambda(\overleftarrow{\partial}) = \Lambda(\partial^\dagger)F = 0\,. \tag{27}$$

These solutions allow us to construct the current $J_\mu$ in the form

$$J_\mu = G\hat{\Gamma}_\mu(\overleftarrow{\partial}, \partial^\dagger)F \tag{28}$$

where

$$\hat{\Gamma}_\mu(\overleftarrow{\partial}, \partial^\dagger) = \sum_{l=1}^{N-1}\sum_{k=0}^{l} \Lambda_{\mu\mu_1\ldots\mu_l} \overleftarrow{\partial}_{\phi_{\mu_1}} \ldots \overleftarrow{\partial}_{\phi_{\mu_k}} \zeta_\mu^- \partial^\dagger_{\phi_{\mu_{k+1}}} \ldots \partial^\dagger_{\phi_{\mu_l}} + \Lambda_\mu \zeta_\mu^-\,. \tag{29}$$

This current is conserved according to the following conservation law:

$$\sum_\mu \partial_{\phi_\mu} J_\mu = 0 \tag{30}$$

where the generalized non-symmetric difference derivative replaces the differential derivatives from the classical formula.

In the proof of this equality we use the modification of the Leibnitz rule (20) and the property of the operator $\Gamma_\mu$ (23):

$$\begin{aligned}
\sum_\mu \partial_{\phi_\mu} J_\mu &= \sum_\mu \partial_{\phi_\mu}\left[G\hat{\Gamma}_\mu(\overleftarrow{\partial}, \partial^\dagger)F\right] \\
&= \sum_\mu \partial_{\phi_\mu} G\left[\sum_{l=1}^{N-1}\sum_{k=0}^{l} \Lambda_{\mu\mu_1\ldots\mu_l}(\overleftarrow{\partial}_{\phi_{\mu_1}})\ldots(\overleftarrow{\partial}_{\phi_{\mu_k}})\zeta_\mu^{-1}\partial^\dagger_{\phi_{\mu_{k+1}}}\ldots\partial^\dagger_{\phi_{\mu_l}} + \Lambda_\mu\zeta_\mu^-\right]F \\
&= \sum_\mu G\left[\sum_{l=1}^{N-1}\sum_{k=0}^{l} \Lambda_{\mu\mu_1\ldots\mu_l}(\overleftarrow{\partial}_{\phi_{\mu_1}})\ldots(\overleftarrow{\partial}_{\phi_{\mu_k}})(\overleftarrow{\partial}_{\phi_\mu} - \partial^\dagger_{\phi_\mu})\partial^\dagger_{\phi_{\mu_{k+1}}}\ldots\partial^\dagger_{\phi_{\mu_l}}\right. \\
&\qquad \left. + \Lambda_\mu(\overleftarrow{\partial}_{\phi_\mu} - \partial^\dagger_{\phi_\mu})\right]F \\
&= G[\Lambda(\overleftarrow{\partial}) - \Lambda(\partial^\dagger)]F = 0\,.
\end{aligned}$$

As the functions $G$ and $F$ fulfil the corresponding equations of motion (27) the current (28) is conserved.

### *3.2. Integrals of motion*

Let us recall the inverse operator for the non-symmetric generalized difference derivative [12, 13]:

$$\int_t^a \mathrm{d}\mu_\phi(\tau)\, u(\tau) = \sum_{n=0}^{\infty} u[\phi^n(t)][\phi^{n+1}(t) - \phi^n(t)] \tag{31}$$

$$\int_b^t \mathrm{d}\mu_\phi(\tau)\, u(\tau) = \sum_{n=1}^{\infty} u[\phi^{-n}(t)][\phi^{-n+1}(t) - \phi^{-n}(t)]\,. \tag{32}$$

These operators obey the fundamental laws of the definite integration:

$$\partial_\phi \int_t^a \mathrm{d}\mu_\phi(\tau)\; u(\tau) = -u(t) \qquad \int_t^a \mathrm{d}\mu_\phi(\tau)\; \partial_\phi u(\tau) = u(a) - u(t) \tag{33}$$

$$\partial_\phi \int_b^t \mathrm{d}\mu_\phi(\tau)\; u(\tau) = u(t) \qquad \int_b^t \mathrm{d}\mu_\phi(\tau)\; \partial_\phi u(\tau) = u(t) - u(b)\,. \tag{34}$$

Provided the definitions are correct that means there exists (finite or not) the limit

$$\lim_{n\to\infty} \phi^n(t) = a$$

corresponding to the construction of the operator (31) or

$$\lim_{n\to\infty} \phi^{-n}(t) = b$$

which allows us to define the operator (32).

We can use now these operators to construct the integrals of motion as

$$Q(x_0) = \int \mathrm{d}\mu_{\phi_1}(x_1)\; \mathrm{d}\mu_{\phi_2}(x_2)\; \mathrm{d}\mu_{\phi_3}(x_3)\; J_0(x_0, \boldsymbol{x})\,. \tag{35}$$

This integral is constant on the time lattice due to the conservation law (30).

Let us note that the construction can easily be extended to models with arbitrary space dimension. In this case we simply integrate over all the spatial dimensions in (35).

## 4. Applications

We shall apply the method derived above to a few examples, namely to the nonlinear difference and generalized difference equation of a mechanical system as well as to the $q$-wave equations being the realizations of the eigenvalue problem of the Casimir operator for the corresponding quantum algebras [3].

### *4.1. Nonlinear difference equation*

Let us start with the following nonlinear difference equation:

$$q_k(t+2h) + q_k(t-2h) = f_k(\mathbf{q}(t)) \qquad k = 1, \ldots, N \tag{36}$$

where $t \in hZ$. This equation can be reformulated to a symplectic map [8], but here we consider only the integrals of motion for the above equation.

According to our method we can rewrite this equation as follows:

$$\bar{D}_h^2 q_k(t) = \frac{f_k(q(t)) + 2q_k(t)}{4h^2} \tag{37}$$

where we have used the symmetric difference derivative and $q(t) = \mathbf{q}(t)\cdot\mathbf{q}(t)$. We construct the integrals of motion by the use of formulae (28), (29):

$$J = \mathbf{q}^{\mathrm{T}}[(\zeta^- + \overleftarrow{\zeta}^{-})\bar{D} - \overleftarrow{\bar{D}}(\zeta^- + \overleftarrow{\zeta}^{-})]\delta\mathbf{q}\,. \tag{38}$$

It is clear that the equation (37) is invariant with respect to translation and rotations. So we choose $\delta$ as $\zeta$ and $R$ where $R$ denotes the $n$-dimensional rotation matrix $R^{\mathrm{T}}R = 1$. When we assume that the functions $f_k$ fulfil the equality:

$$\mathbf{q}^{\mathrm{T}}\zeta\mathbf{f} - \mathbf{f}^{\mathrm{T}}\zeta\mathbf{q} = \mathbf{q}^{\mathrm{T}}R\mathbf{f} - \mathbf{f}^{\mathrm{T}}R\mathbf{q} = 0$$

we obtain the integrals of motion for equation (37):

$$D_h J = 0\,. \tag{39}$$

The same procedure can be applied to more complicated equations that depend explicitly on the lattice variables:

$$\bar{\partial}_\phi^2 q_k(t) = F_k[q(t)]\,. \tag{40}$$

As before we can write down the integrals of motion as

$$J = \mathbf{q}^{\mathrm{T}}[(\zeta^- + \overleftarrow{\zeta}^-)\bar{\partial}_\phi - \overleftarrow{\bar{\partial}_\phi}(\zeta^- + \overleftarrow{\zeta}^-)]R\mathbf{q} \tag{41}$$

provided that the function $\mathbf{F}$ fulfils the equality

$$\mathbf{q}^{\mathrm{T}} R\mathbf{F} - \mathbf{F}^{\mathrm{T}} R\mathbf{q} = 0$$

where $R$ denotes the matrix of rotation.

### 4.2. The q-deformed wave equation in D = 3

We shall apply the procedure to the $q$-deformed wave-equation, which was considered by Floreanini and Vinet [3] in the form

$$\left[(D_t^+)^2 - \mathcal{D}_{x_1}^-\mathcal{D}_{x_2}^-\right]\Phi = 0 \tag{42}$$

We use the original notatation here, where the partial $q$-derivatives act as follows:

$$D_t^+\Phi(t,x_1,x_2) = t^{-1}(-\zeta_t^+ + 1)\Phi(t,x_1,x_2) = (1-q)\partial_q^t\Phi(t,x_1,x_2)$$

$$\mathcal{D}_{x_i}^-\Phi(t,x_1,x_2) = x_i^{-1}(-\zeta_{x_i}^{-2} + 1)\Phi(t,x_1,x_2) = (1-q^{-2})\partial_{q^{-2}}^{x_i}\Phi(t,x_1,x_2)\,.$$

Let us quote the realization of the symmetry operators of this equation obtained in [3], which after reformulation form the quantum algebra:

$$\begin{aligned}
&P_t = D_t^+ \qquad P_1 = \frac{1}{1+q}\mathcal{D}_{x_1}^- \qquad P_2 = \frac{1}{1+q}\mathcal{D}_{x_2}^- \\
&M = \zeta_{x_1}^{+2}\zeta_{x_2}^{-2} \qquad D = q^{\frac{1}{2}}\zeta_t^+\zeta_{x_1}^{+2}\zeta_{x_2}^2 \\
&G_1 = \frac{1}{1-q}(t\mathcal{D}_{x_2}^- - q^2 x_1 D_t^+)\zeta_t^- \qquad G_2 = \frac{1}{1-q}(t\mathcal{D}_{x_1}^- - q^2 x_2 D_t^+)\zeta_t^- \\
&K_1 = \frac{1}{(1-q)^2}(q^{-2}t^2\mathcal{D}_{x_2}^- + x_1^2\mathcal{D}_{x_1}^- - q(1+q)x_1 t D_t^+ - (1-q)x_1)\zeta_t^{-2} \\
&K_2 = \frac{1}{(1-q)^2}(q^{-2}t^2\mathcal{D}_{x_1}^- + x_2^2\mathcal{D}_{x_2}^- - q(1+q)x_2 t D_t^+ - (1-q)x_2)\zeta_t^{-2} \\
&K_0 = \frac{1}{(1-q)^2}(q(t^2 + q^3 x_1 x_2)D_t^+ - t(x_1\mathcal{D}_{x_1}^- + x_2\mathcal{D}_{x_2}^-) + (1-q)t)\zeta_t^{-2}\,.
\end{aligned} \tag{43}$$

Having these transformations of the solution of the wave equation we can construct the conserved current acording to equations (20), (24) and (29). Let us start with the modification of the Leibnitz rule:

$$D_t^+ f\zeta_t^- g = f\left(\overleftarrow{D}_t^+ - (D_t^+)^\dagger\right)g$$

$$\mathcal{D}_{x_i}^- f\zeta_{x_i}^{+2} g = f\left(\overleftarrow{\mathcal{D}}_{x_i}^- - (\mathcal{D}_{x_i}^-)^\dagger\right)g$$

where

$$(D_t^+)^\dagger = -D_t^+\zeta_t^- \qquad (\mathcal{D}_{x_i}^-)^\dagger = -\mathcal{D}_{x_i}^-\zeta_{x_i}^{+2}\,.$$

The operator $\Gamma$ takes the form

$$\Gamma_0 = \overleftarrow{D}_t^+ + (D_t^+)^\dagger \tag{44}$$

$$\Gamma_1 = -\overleftarrow{\mathcal{D}}_{x_2}^- \qquad \Gamma_2 = -(\mathcal{D}_{x_1}^-)^\dagger \,. \tag{45}$$

We now consider the solutions of the wave equation and of its conjugate:

$$\left[(D_t^+)^2 - \mathcal{D}_{x_1}^-\mathcal{D}_{x_2}^-\right] G = 0 \qquad G = \delta\Phi \tag{46}$$

$$\left[(D_t^+)^2 - \mathcal{D}_{x_1}^-\mathcal{D}_{x_2}^-\right]^\dagger F = 0 \tag{47}$$

where $\delta$ denotes transformations leaving the $q$-wave-equation invariant (43) and obtain the conserved current

$$J_\mu = G\hat{\Gamma}_\mu F \tag{48}$$

where

$$\hat{\Gamma}_0 = \overleftarrow{D}_t^+ \zeta_t^- + \zeta_t^-(D_t^+)^\dagger$$

$$\hat{\Gamma}_1 = -\overleftarrow{\mathcal{D}}_{x_2}^- \zeta_{x_1}^{+2} \qquad \hat{\Gamma}_2 = -\zeta_{x_2}^{+2}(\mathcal{D}_{x_1}^-)^\dagger \,.$$

It is clear from the results of section 3 that the following conservation law is fulfilled on-shell ((46), (47)):

$$D_t^+ J_0 + \mathcal{D}_{x_1}^- J_1 + \mathcal{D}_{x_2}^- J_2 = 0 \tag{49}$$

and it can also be rewritten using the standard $q$-derivatives as

$$\partial_q^t J_0 + \partial_{q^2}^{x_1}(-q^{-1} - q^{-2})J_1 + \partial_{q^2}^{x_2}(-q^{-1} - q^{-2})J_2 = 0 \,. \tag{50}$$

The conservation law thus obtained yields the integrals of motion

$$Q(t) = \int \mathrm{d}\mu_1\, \mathrm{d}\mu_2\; J_0(t, x_1, x_2) = \int \mathrm{d}\mu_1\, \mathrm{d}\mu_2\; \delta\Phi[\overleftarrow{D}_t^+ \zeta_t^- + \zeta_t^-(D_t^+)^\dagger]F \tag{51}$$

where $\mathrm{d}\mu_i$ denotes the measure in the definite integral corresponding to $\partial_{q^{-2}}^{x_i}$.

### *4.3. The $q$-deformed wave equation in D = 4*

We shall repeat the construction for the wave equation in four-dimensional space, in light-cone coordinates [3]:

$$\left[D_{x_1}^- D_{x_2}^- - D_{x_3}^+ D_{x_4}^+\right] \Phi = 0 \,. \tag{52}$$

As before, we consider the solutions of (52) and of its conjugate

$$\left[D_{x_1}^- D_{x_2}^- - D_{x_3}^+ D_{x_4}^+\right] \delta\Phi = 0 \qquad \left[D_{x_1}^- D_{x_2}^- - D_{x_3}^+ D_{x_4}^+\right]^\dagger F = 0 \tag{53}$$

where $\delta$ is taken from the set of transformations commuting with the $q$-wave equation [3]:

$$
\begin{aligned}
&P_i = D^-_{x_i} \quad i = 1, 2 \qquad P_i = D^+_{x_i} \quad i = 3, 4\\
&M_1 = \zeta^+_{x_1}\zeta^+_{x_4} \qquad M_2 = \zeta^+_{x_2}\zeta^+_{x_4} \qquad M_3 = \zeta^+_{x_3}\zeta^-_{x_4}\\
&G_1 = \frac{1}{1-q}(x_4 D^-_{x_2} - q x_1 D^+_{x_3})\zeta^+_{x_1} \qquad G_2 = \frac{1}{1-q}(x_3 D^-_{x_2} - q x_1 D^+_{x_4})\zeta^+_{x_1}\\
&G_3 = \frac{1}{1-q}(x_4 D^-_{x_1} - q x_2 D^+_{x_3})\zeta^+_{x_1} \qquad G_4 = \frac{1}{1-q}(x_3 D^-_{x_1} - q x_2 D^+_{x_4})\zeta^+_{x_1}\\
&K_1 = \frac{1}{(1-q)^2}(q^2 x_1 x_2 D^+_{x_4} - x_1 x_3 D^-_{x_1} - x_2 x_3 D^-_{x_2} + q x_3^2 D^+_{x_3} + (1-q)x_3)\zeta^{+2}_{x_1}\\
&K_2 = \frac{1}{(1-q)^2}(q^2 x_1 x_2 D^+_{x_3} - x_1 x_4 D^-_{x_1} - x_2 x_4 D^-_{x_2} + q x_4^2 D^+_{x_4} + (1-q)x_4)\zeta^{+2}_{x_1}\\
&K_3 = \frac{1}{(1-q)^2}(-q x_2 x_3 D^+_{x_3} - q x_2 x_4 D^+_{x_4} + q^{-1} x_3 x_4 D^-_{x_1} + x_2^2 D^-_{x_2} - (1-q)x_2)\zeta^{+2}_{x_1}\\
&K_4 = \frac{1}{(1-q)^2}(-q x_1 x_3 D^+_{x_3} - q x_1 x_4 D^+_{x_4} + q^{-1} x_3 x_4 D^-_{x_2} + x_1^2 D^-_{x_1} - (1-q)x_1)\zeta^{+2}_{x_1}\,.
\end{aligned}
\tag{54}
$$

We obtain the current in the form

$$J_\mu = \delta \Phi \hat{\Gamma}_\mu F \tag{55}$$

where the operator $\hat{\Gamma}$ appears as follows:

$$\hat{\Gamma}_1 = \overleftarrow{D^-_{x_2}}\zeta^+_1 \qquad \hat{\Gamma}_2 = \zeta^+_2 (D^-_{x_1})^\dagger \tag{56}$$

$$\hat{\Gamma}_3 = -\overleftarrow{D^+_{x_4}}\zeta^-_3 \qquad \hat{\Gamma}_4 = -\zeta^-_4 (D^+_{x_3})^\dagger\,. \tag{57}$$

The current thus constructed obeys the conservation law

$$D^-_{x_1} J_1 + D^-_{x_2} J_2 + D^+_{x_3} J_3 + D^+_{x_4} J_4 = 0 \tag{58}$$

which can also be written using $q$-derivatives as

$$\partial^{x_1}_{q^{-1}} J_1 + \partial^{x_2}_{q^{-1}} J_2 + \partial^{x_3}_{q}(-q) J_3 + \partial^{x_4}_{q}(-q) J_4 = 0\,. \tag{59}$$

However, in this case we cannot construct the constants of motion as we are in light-cone coordinates and we do not yet know the formula for changing variables for partial $q$-derivatives.

## 5. Final remarks

In this paper the conservation laws and integrals of motion for a class of equations of motion for discrete and mixed models have been derived.

In order to extend our investigations to arbitrary equations with generalized difference derivatives we should aim at derivation of the Noether-type theorem for such models. So far we have obtained preliminary results [12] for models described by standard difference derivatives. It seems to be interesting and possible to consider as a special case of such a theorem general models invariant with respect to transformations that do not act on the lattice.

As we have shown [13] in discrete mechanics, we obtain equations that include the generalized difference derivative and its conjugate. This class of equations remains to be investigated.

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